Exam Quantum Field Theory
January 20, 2020
Start: 15:00h End: 18:00h

Each sheet with your name and student ID

INSTRUCTIONS: This is a closed-book and closed-notes exam. You are allowed to bring one A4 page written by you, with useful formulas. The exam duration is 3 hours. There is a total of 9 points that you can collect.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also lose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need.

USEFUL FORMULAS

For the energy projectors for spin 1/2 Dirac fermions use the normalization without the factor $1/(2m)$:

$$\sum_{r=1,2} u_r(\vec{p})\bar{u}_r(\vec{p}) = \not{p} + m$$

$$\sum_{r=1,2} v_r(\vec{p})\bar{v}_r(\vec{p}) = \not{p} - m$$

$$\{\gamma_5, \gamma^\mu\} = 0, \quad (\gamma^0)^2 = \mathbb{1}, \quad \gamma_5^2 = \mathbb{1}, \quad \gamma_5^\dagger = \gamma_5, \quad \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})$$

$$\gamma^\mu\gamma_\mu = 4\mathbb{1} \quad \gamma^\mu\not{p}\gamma_\mu = -2\not{p} \quad \not{k}\not{p}\not{k} = 2(pk)\not{k} - k^2\not{p}$$

$$\mathrm{Tr}(\gamma_5\gamma^\mu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$$

1. (3 points total) The Lagrangian density:

$$\mathcal{L} = \partial_\mu \chi^\dagger \partial^\mu \chi - m^2 \chi^\dagger \chi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + \lambda \chi^\dagger \chi \phi \,,$$

describes the interaction with coupling λ of a real scalar field ϕ with mass M and a complex scalar field χ with mass m.

a) **[1.5 points]** Write the path integral for this theory in presence of external sources, $Z[J_0, J, J^\dagger]$. Work out its perturbative expansion in powers of the coupling λ, or equivalently in powers of the interaction Lagrangian $\mathcal{L}_I = \lambda \chi^\dagger \chi \phi$, where the latter can be rewritten as a functional of the derivatives w.r.t. the external sources acting on the path integral of the free theory, $Z_0[J_0, J, J^\dagger]$. **Show your work**

b) **[1.5 points]** Derive the Feynman rule (in momentum space) for the vertex and draw the corresponding Feynman diagram. **Show your work**

2. (3 points total) Consider the QED Lagrangian density with the addition of the Pauli term:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c\bar{\psi}\sigma^{\mu\nu}\psi\, F_{\mu\nu}$$

with $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$, m the fermion mass, $F_{\mu\nu}$ the fully antisymmetric electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, c the coupling of the Pauli term and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$.

a) **[1 points]** Find how each term in the Lagrangian transforms under the global chiral transformation:

$$\begin{aligned} \psi'(x) &= e^{i\alpha\gamma_5}\,\psi(x) \\ \bar{\psi}'(x) &= \bar{\psi}(x)e^{i\alpha\gamma_5} \\ A'_\mu(x) &= A_\mu(x) \end{aligned}$$

with α a real parameter of the transformation. **Show your work**

b) **[1.5 points]** Show that the vector current $J^V_\mu = \bar{\psi}\gamma_\mu\psi$ is conserved on the equation of motion for the electromagnetic field, or, equivalently, on the equations of motion for ψ and $\bar{\psi}$. **Show your work**

c) **[0.5 points]** According to Noether's theorem there is a symmetry associated to the conservation of J^V_μ. What is this symmetry? Write explicitly the corresponding transformations of the fields. **Show your work**

3. (3 points total)

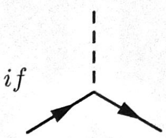
Consider the lepton pair production process $e^-e^+ \to \mu^-\mu^+$ in the Yukawa theory. The free Lagrangian density of the Yukawa theory reads:

$$\mathcal{L}_0 = \sum_{l=e,\mu,\tau} \bar{\psi}_l \left(i\partial\!\!\!/ - m_l \right) \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 \,,$$

with ϕ a real (pseudo)scalar field with mass M, ψ_l the fermion field for lepton l with mass m_l, and $l = e$ for the electron (positron), $l = \mu$ for the muon (antimuon) and $l = \tau$ for the tau (antitau).

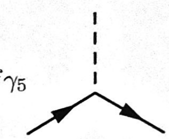
If ϕ is a scalar under Lorentz transformations, then the Yukawa interaction with degenerate couplings reads:

$$\mathcal{L}_I = f \sum_{l=e,\mu,\tau} \bar{\psi}_l \psi_l \phi \,, \qquad \text{Feynman rule:} \quad if$$



If ϕ is a pseudoscalar under Lorentz transformations, then the Yukawa interaction with degenerate couplings contains a γ_5 and reads:

$$\mathcal{L}_{I5} = if \sum_{l=e,\mu,\tau} \bar{\psi}_l \gamma_5 \psi_l \phi \,, \qquad \text{Feynman rule:} \quad f\gamma_5$$



a) **[0.5 points]** Draw the Feynman diagrams that contribute to $e^-(p_1)e^+(p_2) \to \mu^-(p_1')\mu^+(p_2')$ to leading order in f for the case with interaction $\mathcal{L}_I$ and the case with interaction $\mathcal{L}_{I5}$.

b) **[1.5 points]** Derive the corresponding unpolarized squared amplitude in both cases, thus:

$$X = \left(\mathcal{A}^\dagger \mathcal{A} \right)_{\text{unpol.}}$$

$$X_5 = \left(\mathcal{A}_5^\dagger \mathcal{A}_5 \right)_{\text{unpol.}}$$

in terms of the four-momenta of $e^\pm$ and $\mu^\pm$ and in the limit of vanishing electron (positron) mass $m_e = 0$. **Show your work**

c) **[1 points]** Compute the corresponding cross sections σ and σ_5 in the centre of mass frame and show that

$$\sigma = \left(1 - \frac{4m_\mu^2}{s} \right) \sigma_5 \qquad (1)$$

where both σ and σ_5 vanish at threshold, i.e. for $s = 4m_\mu^2$ with $s = (p_1 + p_2)^2$. Which cross section is larger above threshold?

Hints:

- Use the sum over fermion spins given on page 1, i.e. without the normalization factor $1/(2m)$.
- The differential cross section in the centre of mass (CoM) frame for a $2 \to 2$ particle process reads:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}'|}{|\vec{p}|} X$$

  with $\vec{p}$ ($\vec{p}'$) the single particle tri-momentum in the initial (final) state in the CoM and X is the unpolarized squared amplitude.